Bull. Math. Sci. (2018) 8:33–47
https://doi.org/10.1007/s13373-016-0098-x

An improved regularity criterion for the Navier–Stokes equations in terms of one directional derivative of the velocity field

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Received: 4 December 2015 / Revised: 24 December 2016 / Accepted: 28 December 2016 /
Published online: 4 January 2017

Abstract In this paper, we establish a new multiplicative Sobolev inequality. As applications, we refine and extend the results in Kukavica and Ziane (J Math Phys 48:065203, 2007) and Cao (Discrete Contin Dyn Syst 26:1141–1151, 2010) simultaneously.

Keywords Regularity criteria · Navier–Stokes equations · Multiplicative Sobolev inequality

Mathematics Subject Classification 35B65 · 35Q30 · 76D03

1 Introduction

The homogeneous incompressible fluid flow is governed by the following Navier–Stokes equations (NSE):

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, π is a scalar pressure, $\boldsymbol{u}_0$ is the prescribed initial velocity field satisfying the compatibility condition $\nabla \cdot \boldsymbol{u}_0 = 0$, and

Communicated by Neil Trudinger.

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$$\partial_t \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad (\boldsymbol{u} \cdot \nabla) = \sum_{i=1}^{3} u_i \partial_i.$$

The global existence of a weak solution to the evolutionary NSE (1) has been long established by Leray [19] and Hopf [9]; however, the issue of its regularity and uniqueness remains open up to now. Pioneered by Serrin [26], we began studying the regularity criterion for the NSE (1); that is, to find some sufficient condition to ensure the smoothness of the solution. The classical Prodi-Serrin conditions (see [8,24,26]) says that if

$$\boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \le q \le \infty, \tag{2}$$

then the solution is regular on $(0, T)$.

This was be generalized by Beirão da Veiga [1] by considering the velocity gradient or vorticity,

$$\nabla \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} \le q \le \infty. \tag{3}$$

Notice that the case $q \in \left[\frac{3}{2}, 3\right)$ follows directly from (2) and the Sobolev inequality.

In view of the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$, it is natural to ask whether or not we can reduce (2) and (3) to its partial components. One way is to consider regularity criteria involving only one velocity component, which were done in [3,11,16,20,34, 36]. Another way is to study the possible components reduction of $\nabla \boldsymbol{u}$ to ∇u_3, see [16,23,27,35,36]; or to $\partial_3 \boldsymbol{u}$, see [2,17,21,22]. In [22], Penel–Pokorný showed that if

$$\partial_3 \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \le q \le \infty, \tag{4}$$

then the solution is smooth. This was improved by Kukavica–Ziane [17] to be

$$\partial_3 \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{9}{4} \le q \le 3. \tag{5}$$

Notice that the range of q is not of full range $\left(\frac{3}{2}, \infty\right]$. The reason is that in [17], the estimate of I_3 needs to be reconciled with the the estimate of K. Furthermore, this method was adjusted by Penel–Pokorný [21] to get an anisotropic criterion. For readers interested in this topic, please refer to [12–15,30] for recent progresses on regularity criteria of the MHD equations, which contains system (1) as a subsystem.

Later on, Cao [2] employed multiplicative Sobolev inequalities

$$1 \le q < \infty \Rightarrow \|f\|_{L^{3q}} \le C \|\partial_1 f\|_{L^2}^{\frac{1}{3}} \|\partial_2 f\|_{L^2}^{\frac{1}{3}} \|\partial_3 f\|_{L^q}^{\frac{1}{3}} \tag{6}$$

and

$$1 \le q < \infty \Rightarrow \|f\|_{L^{5q}} \le C \left\|\partial_1(f^2)\right\|_{L^2}^{\frac{1}{5}} \left\|\partial_2(f^2)\right\|_{L^2}^{\frac{1}{5}} \|\partial_3 f\|_{L^q}^{\frac{1}{5}} \tag{7}$$

to get the following extended regularity condition[1]

$$\partial_3 \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{27}{16} \le q \le \frac{5}{2}. \tag{8}$$

Notice that the lower and upper bounds of q in (8) both are less than those in (5) respectively. Consequently, our best knowledge in this direction is the following sufficient condition

$$\partial_3 \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{27}{16} \le q \le 3. \tag{9}$$

Some of them was proved in [17], while other parts could only be seen [2].

In this paper, we shall further generalize (7), and improve (5) and (8) simultaneously. We will show that the condition

$$\partial_3 \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3\sqrt{37}}{4} - 3 \le q \le 3 \tag{10}$$

could ensure the regularity of the solution. Noting

$$\frac{3\sqrt{37}}{4} \approx 1.56207 < 1.6875 = \frac{27}{16},$$

we are much closer to the end point $\dfrac{3}{2}$.

Before stating the precise result, let us recall the weak formulation of (1), see [7,18,25,28] for instance.

Definition 1 Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = 0$, $T > 0$. A measurable $\mathbb{R}^3$-valued function $\boldsymbol{u}$ defined in $[0, T] \times \mathbb{R}^3$ is said to be a weak solution to (1) if

1. $\boldsymbol{u} \in L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^2(0, T; H^1(\mathbb{R}^3));$
2. $(1)_1$ and $(1)_2$ hold in the sense of distributions, i.e.,

$$\int_0^t \int_{\mathbb{R}^3} \boldsymbol{u} \cdot [\partial_t \boldsymbol{\phi} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{\phi}]\, \mathrm{d}x\, \mathrm{d}s + \int_{\mathbb{R}^3} \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(0)\, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{u} : \nabla \boldsymbol{\phi}\, \mathrm{d}x\, \mathrm{d}t,$$

[1] In [2, Theorem 2.1], the author claims that (8) is valid for all $q \in [27/16, \infty]$; however, only the case $q \in [27/16, 5/2]$ could be verified in his paper, see the inequality just before (32).

for each $\boldsymbol{\phi} \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{\phi} = 0$, where $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$ for 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$, and

$$\int_0^T \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = 0,$$

for each $\psi \in C_c^\infty(\mathbb{R}^3 \times [0,T))$;

3. the energy inequality, that is,

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \boldsymbol{u}(s)\|_{L^2}^2 \, \mathrm{d}s \leq \|\boldsymbol{u}_0\|_{L^2}^2, \quad 0 \leq t \leq T.$$

Now, our main result reads

Theorem 2 *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = 0$, $T > 0$. Assume that $\boldsymbol{u}$ is a weak solution to (1) on $[0,T]$ with initial data $\boldsymbol{u}_0$. If*

$$\partial_3 \boldsymbol{u} \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3\sqrt{37}}{4} - 3 \leq q \leq 3, \tag{11}$$

then the solution $\boldsymbol{u}$ is smooth in $(0,T] \times \mathbb{R}^3$.

The proof of Theorem 2 will be given in Sect. 2. Before doing that, let us state our notations used throughout the paper, and prove a multiplicative Sobolev inequality.

For simplicity of presentation, we do not distinguish between the spaces X and their N-dimensional vector analogs X^N (e.g., $N = 3$ for $\boldsymbol{u} \in L^2(\mathbb{R}^3)$, $N = 9$ for $\nabla \boldsymbol{u} \in L^2(\mathbb{R}^3)$); however, all vector- and tensor-valued functions are printed boldfaced. A constant C may change from line to line, depending only on the initial data or the norms that we have controlled. We denote by

$$\boldsymbol{u}_h = (u_1, u_2), \quad \nabla_h = (\partial_1, \partial_2), \quad \triangle_h = \partial_1 \partial_1 + \partial_2 \partial_2.$$

Generalizing (7) in [2], we have the following

Lemma 3 *For each $1 \leq q < \infty$, $0 < \lambda < \infty$, there exists some constant C such that for each $f \in C_c^\infty(\mathbb{R}^3)$,*

$$\|f\|_{L^{(2\lambda+1)q}} \leq C \, \|\partial_i f\|_{L^q}^{\frac{1}{2\lambda+1}} \left\|\partial_j(|f|^\lambda)\right\|_{L^2}^{\frac{1}{2\lambda+1}} \left\|\partial_k(|f|^\lambda)\right\|_{L^2}^{\frac{1}{2\lambda+1}},$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$.

Proof By Newton–Leibniz formula, we have

$$
\begin{aligned}
|f|^{2\lambda(q-1)+q} &\le C\int_{\mathbb{R}} |f|^{(2\lambda+1)(q-1)}|\partial_i f|\,\mathrm{d}x_i,\\
|f|^{\lambda(q+1)+\frac{q}{2}} &= \left(|f|^{\lambda}\right)^{q+1+\frac{q}{2\lambda}} \le C\int_{\mathbb{R}} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_j(|f|^{\lambda})|\,\mathrm{d}x_j,\\
|f|^{\lambda(q+1)+\frac{q}{2}} &\le C\int_{\mathbb{R}} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_k(|f|^{\lambda})|\,\mathrm{d}x_k.
\end{aligned}
$$

Taking the sqrt of the multiplication of the above inequalities, we deduce

$$
\begin{aligned}
|f|^{(2\lambda+1)q} \le& \left[\int_{\mathbb{R}} |f|^{(2\lambda+1)(q-1)}|\partial_i f|\,\mathrm{d}x_i\right]^{\frac{1}{2}}\\
&\times\left[\int_{\mathbb{R}} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_j(|f|^{\lambda})|\,\mathrm{d}x_j\right]^{\frac{1}{2}}\cdot\left[\int_{\mathbb{R}} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_k(|f|^{\lambda})|\,\mathrm{d}x_k\right]^{\frac{1}{2}}
\end{aligned}
$$

Integrating in the x_i variable and applying Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} |f|^{(2\lambda+1)q}\,\mathrm{d}x_i \le C\left[\int_{\mathbb{R}} |f|^{(2\lambda+1)(q-1)}|\partial_i f|\,\mathrm{d}x_i\right]^{\frac{1}{2}}\\
&\cdot\left[\int_{\mathbb{R}^2} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_j(|f|^{\lambda})|\,\mathrm{d}x_i\,\mathrm{d}x_j\right]^{\frac{1}{2}}\cdot\left[\int_{\mathbb{R}^2} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_k(|f|^{\lambda})|\,\mathrm{d}x_i\,\mathrm{d}x_k\right]^{\frac{1}{2}}
\end{aligned}
$$

Successively, integrating in the x_j and x_k variables and applying Hölder inequality, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |f|^{(2\lambda+1)q}\,\mathrm{d}x \le C\left[\int_{\mathbb{R}^3} |f|^{(2\lambda+1)(q-1)}|\partial_i f|\,\mathrm{d}x\right]^{\frac{1}{2}}\\
&\cdot\left[\int_{\mathbb{R}^3} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_j(|f|^{\lambda})|\,\mathrm{d}x\right]^{\frac{1}{2}}\cdot\left[\int_{\mathbb{R}^3} |f|^{\frac{(2\lambda+1)q}{2}}|\partial_k(|f|^{\lambda})|\,\mathrm{d}x\right]^{\frac{1}{2}}.
\end{aligned}
$$

Invoking Hölder inequality again, we find

$$
\begin{aligned}
\|f\|_{L^{(2\lambda+1)q}}^{(2\lambda+1)q} \le& C\left[\|f\|_{L^{(2\lambda+1)q}}^{(2\lambda+1)(q-1)}\,\|\partial_i f\|_{L^q}^{\frac{1}{2}}\right]^{\frac{1}{2}}\\
&\cdot\left[\|f\|_{L^{(2\lambda+1)q}}^{\frac{(2\lambda+1)q}{2}}\,\left\|\partial_j(|f|^{\lambda})\right\|_{L^2}\right]^{\frac{1}{2}}\left[\|f\|_{L^{(2\lambda+1)q}}^{\frac{(2\lambda+1)q}{2}}\,\left\|\partial_k(|f|^{\lambda})\right\|_{L^2}\right]^{\frac{1}{2}}\\
\le& C\,\|f\|_{L^{(2\lambda+1)q}}^{(2\lambda+1)\left(q-\frac{1}{2}\right)}\,\|\partial_i f\|_{L^q}^{\frac{1}{2}}\,\left\|\partial_j(|f|^{\lambda})\right\|_{L^2}^{\frac{1}{2}}\,\left\|\partial_k(|f|^{\lambda})\right\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Dividing both sides by $\|f\|_{L^{(2\lambda+1)q}}^{(2\lambda+1)\left(q-\frac{1}{2}\right)}$, we finished the proof of Lemma 3.

2 Proof of Theorem 2

In this section, we shall prove Theorem 2.

For any $\varepsilon \in (0, T)$, due to the fact that $\nabla \boldsymbol{u} \in L^2(0, T; L^2(\mathbb{R}^3))$, we may find a $\delta \in (0, \varepsilon)$, such that $\nabla \boldsymbol{u}(\delta) \in L^2(\mathbb{R}^3)$. Take this $\boldsymbol{u}(\delta)$ as initial data, there exists an $\tilde{\boldsymbol{u}} \in C([\delta, \Gamma^*), H^1(\mathbb{R}^3)) \cap L^2(0, \Gamma^*; H^2(\mathbb{R}^3))$, where $[\delta, \Gamma^*)$ is the life span of the unique strong solution, see [28]. Moreover, $\tilde{\boldsymbol{u}} \in C^\infty(\mathbb{R}^3 \times (\delta, \Gamma^*))$. According to the uniqueness result, $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $[\delta, \Gamma^*)$. If $\Gamma^* \geq T$, we have already that $\boldsymbol{u} \in C^\infty(\mathbb{R}^3 \times (0, T))$, due to the arbitrariness of $\varepsilon \in (0, T)$. In case $\Gamma^* < T$, our strategy is to show that $\|\nabla \boldsymbol{u}_h(t)\|_2$ remains uniform bounded as $t \nearrow \Gamma^*$. By [33, Proposition 1.1], we have $\|\nabla \boldsymbol{u}(t)\|_2$ remains uniform bounded as $t \nearrow \Gamma^*$. The standard continuation argument then yields that $[\delta, \Gamma^*)$ could not be the maximal interval of existence of $\tilde{\boldsymbol{u}}$, and consequently $\Gamma^* \geq T$. This concludes the proof.

By (11), we may find a $\Gamma < \Gamma^*$ such that

$$\|\nabla \boldsymbol{u}(\Gamma)\|_{L^2} \leq C, \quad \left(\int_\Gamma^{\Gamma^*} \|\partial_3 \boldsymbol{u}\|_{L^q}^{\frac{2q}{2q-3}} \,\mathrm{d}t\right)^{\frac{2q-3}{2q}} < \tilde{\varepsilon}, \tag{12}$$

where $0 < \tilde{\varepsilon} \ll 1$ will be determined later on.

For convenience, we rewrite the NSE (1) as

$$\begin{aligned} \partial_t \boldsymbol{u}_h + (\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h + u_3 \partial_3 \boldsymbol{u}_h - \triangle_h \boldsymbol{u}_h - \partial_3\partial_3 \boldsymbol{u}_h + \nabla_h \pi &= \boldsymbol{0}, \\ \partial_t u_3 + (\boldsymbol{u}_h \cdot \nabla) u_3 + u_3 \partial_3 u_3 - \triangle_h u_3 - \partial_3 \partial_3 u_3 + \partial_3 \pi &= 0, \\ \nabla_h \cdot \boldsymbol{u}_h + \partial_3 u_3 = 0. & \end{aligned} \tag{13}$$

2.1 H^1 estimate

Taking the inner product of $(13)_1$ with $-\triangle \boldsymbol{u}_h$ and $(1)_1$ with $-\partial_3\partial_3 \boldsymbol{u}$ in $L^2(\mathbb{R}^3)$ respectively, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla \boldsymbol{u}_h\|_{L^2}^2 + \|\partial_3 \boldsymbol{u}\|_{L^2}^2\right] + \left[\|\triangle \boldsymbol{u}_h\|_{L^2}^2 + \|\nabla \partial_3 \boldsymbol{u}\|_{L^2}^2\right] \\ &= \int_{\mathbb{R}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_h] \cdot \triangle \boldsymbol{u}_h \,\mathrm{d}x + \int_{\mathbb{R}^3} \nabla_h \pi \cdot \triangle \boldsymbol{u}_h \,\mathrm{d}x + \int_{\mathbb{R}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \partial_3\partial_3 \boldsymbol{u} \,\mathrm{d}x \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{14}$$

By [17, Lemma 2.2],

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} [(\boldsymbol{u}_h \cdot \nabla_h)\boldsymbol{u}_h] \cdot \triangle_h \boldsymbol{u}_h \,\mathrm{d}x + \int_{\mathbb{R}^3} u_3 \partial_3 \boldsymbol{u}_h \cdot \triangle \boldsymbol{u}_h \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \partial_3 u_3 |\nabla_h \boldsymbol{u}_h|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 \,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 \,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3} u_3\partial_3\boldsymbol{u}_h\cdot\triangle\boldsymbol{u}_h\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\partial_3\boldsymbol{u}|\cdot|\nabla_h\boldsymbol{u}_h|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|u_3|\cdot|\partial_3\boldsymbol{u}|\cdot|\triangle\boldsymbol{u}_h|\,\mathrm{d}x. \qquad (15)
\end{aligned}
$$

To simplify I_2, we take the divergence of $(1)_1$ to get

$$
\begin{aligned}
-\triangle\pi &= \sum_{i,j=1}^{3}\partial_i(u_j\partial_j u_i)\\
&= \sum_{i,j=1}^{2}\partial_i u_j\partial_j u_i \quad \text{(the case: } i\neq 3,\ j\neq 3)\\
&\quad+\sum_{j=1}^{2}\partial_3 u_j\partial_j u_3+u_j\partial_j\partial_3 u_3 \quad \text{(the case: } i=3,\ j\neq 3)\\
&\quad+\sum_{i=1}^{3}\partial_i u_3\partial_3 u_i \quad \text{(the case: } j=3) \qquad (16)
\end{aligned}
$$

and thus

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3}\triangle\pi\cdot\nabla_h\boldsymbol{u}_h\,\mathrm{d}x=\int_{\mathbb{R}^3}\triangle\pi\cdot\partial_3 u_3\,\mathrm{d}x\\
&= \sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_i u_j\partial_j u_i\partial_3 u_3\,\mathrm{d}x+\sum_{j=1}^{2}\int_{\mathbb{R}^3}\left(\partial_3 u_j\partial_j u_3+u_j\partial_j\partial_3 u_3\right)\partial_3 u_3\,\mathrm{d}x\\
&\quad+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3\partial_3 u_i\partial_3 u_3\,\mathrm{d}x\\
&\le \int_{\mathbb{R}^3}|\partial_3 u_3|\cdot|\nabla_h\boldsymbol{u}_h|^2\,\mathrm{d}x-\sum_{j=1}^{2}\int_{\mathbb{R}^3}u_3(\partial_3\partial_j u_j\partial_3 u_3+\partial_3 u_j\partial_j\partial_3 u_3)\,\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^3}(\partial_3 u_3)^2\,\mathrm{d}x-\sum_{i=1}^{3}\int_{\mathbb{R}^3}u_3\partial_3 u_i\partial_i\partial_3 u_3\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\partial_3 u_3|\cdot|\nabla_h\boldsymbol{u}_h|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|u_3|\cdot|\partial_3\boldsymbol{u}|\cdot|\nabla\partial_3\boldsymbol{u}|\,\mathrm{d}x. \qquad (17)
\end{aligned}
$$

Finally, integrating by parts yields

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}^3}[(\partial_3\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_3\boldsymbol{u}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}[(\partial_3\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{u}_h]\cdot\partial_3\boldsymbol{u}_h+[(\partial_3\boldsymbol{u}_h\cdot\nabla_h)u_3]\cdot\partial_3 u_3+\partial_3 u_3\partial_3\boldsymbol{u}\cdot\partial_3\boldsymbol{u}\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\partial_3\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}_h|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}|u_3|\cdot|\partial_3\boldsymbol{u}|\cdot|\nabla\partial_3\boldsymbol{u}|\,\mathrm{d}x. \qquad (18)
\end{aligned}
$$

Gathering (15), (17) and (18) into (14), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla \boldsymbol{u}_h\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^2}^2\right]+\left[\|\triangle \boldsymbol{u}_h\|_{L^2}^2+\|\nabla\partial_3\boldsymbol{u}\|_{L^2}^2\right]\\
&\quad\leq C\int_{\mathbb{R}^3}|\partial_3\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}_h|^2\,\mathrm{d}x+C\int_{\mathbb{R}^3}|u_3|\cdot|\partial_3\boldsymbol{u}|\cdot(|\triangle\boldsymbol{u}_h|+|\nabla\partial_3\boldsymbol{u}|)\,\mathrm{d}x\\
&\quad\equiv J_1+J_2+J_3.
\end{aligned}\tag{19}$$

By Hölder and Gagliardo-Nirenberg inequalities,

$$\begin{aligned}
J_1&\leq C\,\|\partial_3\boldsymbol{u}\|_{L^q}\,\|\nabla\boldsymbol{u}_h\|_{L^{\frac{2q}{q-1}}}^2\\
&\leq C\,\|\partial_3\boldsymbol{u}\|_{L^q}\,\|\nabla\boldsymbol{u}_h\|_{L^2}^{\frac{2q-3}{q}}\,\left\|\nabla^2\boldsymbol{u}_h\right\|_{L^2}^{\frac{3}{q}}\\
&\leq C\,\|\partial_3\boldsymbol{u}\|_{L^q}\,\|\nabla\boldsymbol{u}_h\|_{L^2}^{\frac{2q-3}{q}}\,\|\triangle\boldsymbol{u}_h\|_{L^2}^{\frac{3}{q}}
\end{aligned}\tag{20}$$

For J_2, we first use Hölder inequality with

$$1\leq a,b\leq\infty,\quad \frac{1}{a}+\frac{1}{b}=\frac{1}{2}\tag{21}$$

to estimate as

$$J_2\leq C\,\|u_3\|_{L^a}\,\|\partial_3\boldsymbol{u}\|_{L^b}\,\|(\triangle\boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}\,,$$

then invoke the interpolation and Gagliardo-Nirenberg inequalities to bound as

$$J_2\leq C\,\|u_3\|_{L^{2\lambda}}^{1-\vartheta_1}\,\|u_3\|_{L^{(2\lambda+1)q}}^{\vartheta_1}\cdot\|\partial_3\boldsymbol{u}\|_{L^q}^{1-\vartheta_2}\,\|\nabla\partial_3\boldsymbol{u}\|_{L^2}^{\vartheta_2}\cdot\|(\triangle\boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}\,,$$

where

$$\frac{1}{a}=\frac{1-\vartheta_1}{2\lambda}+\frac{\vartheta_1}{(2\lambda+1)q},\quad \frac{1}{b}=\frac{1-\vartheta_2}{q}+\vartheta_2\left(-\frac{1}{3}+\frac{1}{2}\right),\quad 0\leq\vartheta_1,\vartheta_2\leq 1,\tag{22}$$

and $\lambda\geq\frac{3}{2}$ will be specified later on.

By Lemma 3,

$$\begin{aligned}
J_2&\leq C\,\left\||u_3|^\lambda\right\|_{L^2}^{\frac{1-\vartheta_1}{\lambda}}\cdot\left\|\nabla_h(|u_3|^\lambda)\right\|_{L^2}^{\frac{2\vartheta_1}{2\lambda+1}}\,\|\partial_3u_3\|_{L^q}^{\frac{\vartheta_1}{2\lambda+1}}\cdot\|\partial_3\boldsymbol{u}\|_{L^q}^{1-\vartheta_2}\,\|(\triangle\boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}^{1+\vartheta_2}\\
&\leq C\,\left\||u_3|^\lambda\right\|_{L^2}^{\frac{1-\vartheta_1}{\lambda}}\,\left\|\nabla_h(|u_3|^\lambda)\right\|_{L^2}^{\frac{2\vartheta_1}{2\lambda+1}}\,\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\,\|(\triangle\boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}^{1+\vartheta_2}
\end{aligned}\tag{23}$$

Plugging (20) and (23) into (19), we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla \boldsymbol{u}_h\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^2}^2\right]+\left[\|\triangle \boldsymbol{u}_h\|_{L^2}^2+\|\nabla\partial_3\boldsymbol{u}\|_{L^2}^2\right]\\
&\quad\le C\,\|\partial_3\boldsymbol{u}\|_{L^q}\,\|\nabla \boldsymbol{u}_h\|_{L^2}^{\frac{2q-3}{q}}\,\|\triangle \boldsymbol{u}_h\|_{L^2}^{\frac{3}{q}}\\
&\qquad+C\,\left\||u_3|^\lambda\right\|_{L^2}^{\frac{1-\vartheta_1}{\lambda}}\,\left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{2\vartheta_1}{2\lambda+1}}\,\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\,\|(\triangle \boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}^{1+\vartheta_2}.
\end{aligned}$$

Integrating in time and denoting by

$$\mathcal{J}^2(t)=\sup_{\Gamma\le\tau\le t}\left[\|\nabla \boldsymbol{u}_h\|_{L^2}^2+\|\partial_3\boldsymbol{u}\|_{L^2}^2\right]+\int_\Gamma^t\left[\|\triangle \boldsymbol{u}_h\|_{L^2}^2+\|\nabla\partial_3\boldsymbol{u}\|_{L^2}^2\right]\mathrm{d}\tau, \tag{24}$$

$$\mathcal{L}^2(t)=\sup_{\Gamma\le\tau\le t}\left\||u_3|^\lambda\right\|_{L^2}^2+\int_\Gamma^t\left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^2\,\mathrm{d}\tau,\quad \Gamma\le t<\Gamma^*, \tag{25}$$

we deduce

$$\begin{aligned}
\mathcal{J}^2(t)&\le\|\nabla \boldsymbol{u}_h(\Gamma)\|_{L^2}^2+\|\partial_3\boldsymbol{u}(\Gamma)\|_{L^2}^2+C\mathcal{J}^{\frac{2q-3}{q}}(t)\cdot\int_\Gamma^t\|\partial_3\boldsymbol{u}\|_{L^q}\,\|\triangle \boldsymbol{u}_h\|_{L^2}^{\frac{3}{q}}\,\mathrm{d}\tau\\
&\quad+C\mathcal{L}^{\frac{1-\vartheta_1}{\lambda}}(t)\cdot\int_\Gamma^t\left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{2\vartheta_1}{2\lambda+1}}\,\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\,\|(\triangle \boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}^{1+\vartheta_2}\,\mathrm{d}\tau\\
&\le C+C\mathcal{J}^{\frac{2q-3}{q}}(t)\cdot\left(\int_\Gamma^t\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{2q}{2q-3}}\,\mathrm{d}\tau\right)^{\frac{2q-3}{2q}}\left(\int_\Gamma^t\|\triangle \boldsymbol{u}_h\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{3}{2q}}\\
&\quad+C\mathcal{L}^{\frac{1-\vartheta_1}{\lambda}}(t)\cdot\left(\int_\Gamma^t\left\|\nabla_h(|u_3|^\lambda)\right\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{\vartheta_1}{2\lambda+1}}\\
&\quad\cdot\left(\int_\Gamma^t\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{2q}{2q-3}}\,\mathrm{d}\tau\right)^{\frac{2q-3}{2q}\left(\frac{\vartheta_1}{\lambda+1}+1-\vartheta_2\right)\mathrm{d}\tau}\cdot\left(\int_\Gamma^t\|(\triangle \boldsymbol{u}_h,\nabla\partial_3\boldsymbol{u})\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1+\vartheta_2}{2}}\\
&\le C+C\tilde{\varepsilon}\mathcal{J}^2(t)+C\mathcal{L}^{\frac{1-\vartheta_1}{\lambda}}(t)\cdot\mathcal{L}^{\frac{2\vartheta_1}{2\lambda+1}}(t)\cdot\tilde{\varepsilon}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\cdot\mathcal{J}^{1+\vartheta_2}(t)\\
&\le C+C\tilde{\varepsilon}\mathcal{J}^2(t)+C\tilde{\varepsilon}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\mathcal{J}^{1+\vartheta_2}(t)\mathcal{L}^{\frac{1-\vartheta_1}{\lambda}+\frac{2\vartheta_1}{2\lambda+1}}(t),
\end{aligned}\tag{26}$$

where Hölder inequality with

$$\frac{\vartheta_1}{2\lambda+1}+\frac{2q-3}{2q}\left(\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2\right)+\frac{1+\vartheta_2}{2}=1 \tag{27}$$

is used.

2.2 $L^{2\lambda}$ estimate

Taking the inner product of (13)$_3$ with $2\lambda|u_3|^{2\lambda-2}u_3$ in $L^2(\mathbb{R}^3)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\||u_3|^{\lambda}\right\|_{L^2}^2 + C(\lambda)\left\|\nabla(|u_3|)^{\lambda}\right\|_{L^2}^2 = 2\lambda\int_{\mathbb{R}^3}\partial_3\pi\,|u_3|^{2\lambda-2}u_3\,\mathrm{d}x \equiv L. \tag{28}$$

To process L, we derive from (16) that

$$\begin{aligned}
-\triangle\partial_3\pi &= \sum_{i,j=1}^{3}\partial_i\partial_j(\partial_3u_iu_j + u_i\partial_3u_j) = 2\sum_{i,j=1}^{3}\partial_i\partial_j(u_i\partial_3u_j)\\
&= 2\sum_{i=1}^{2}\sum_{j=1}^{3}\partial_i\partial_j(u_i\partial_3u_j) + 2\sum_{j=1}^{3}\partial_3\partial_j(u_3\partial_3u_j)\\
\Rightarrow \partial_3\pi &= 2\sum_{i=1}^{2}\sum_{j=1}^{3}\mathcal{R}_i\mathcal{R}_j(u_i\partial_3u_j) + 2\sum_{j=1}^{3}\mathcal{R}_3\mathcal{R}_j(u_3\partial_3u_j) \equiv \pi_1+\pi_2,
\end{aligned} \tag{29}$$

where $\mathcal{R}_i = \dfrac{\partial_i}{\sqrt{-\triangle}}$ is the Riesz transformation, which is bounded from $L^r(\mathbb{R}^3))$ to itself for $1 < r < \infty$.

In view of (29),

$$\begin{aligned}
L &= 2\lambda\int_{\mathbb{R}^3}(\pi_1+\pi_2)|u_3|^{2\lambda-2}u_3\,\mathrm{d}x\\
&\leq C\left\|\boldsymbol{u}_h\right\|_{L^c}\left\|\partial_3\boldsymbol{u}\right\|_{L^q}\left\|u_3\right\|_{L^d}^{2\lambda-1} + C\left\|u_3\right\|_{L^{\frac{2\lambda q}{q-1}}}\left\|\partial_3\boldsymbol{u}\right\|_{L^q}\left\|u_3\right\|_{L^{\frac{2\lambda q}{q-1}}}^{2\lambda-1},
\end{aligned}$$

where

$$1\leq c,d\leq\infty,\quad \frac{1}{c}+\frac{1}{q}+\frac{2\lambda-1}{d}=1. \tag{30}$$

By Gagliardo-Nirenberg and interpolation inequalities,

$$\begin{aligned}
L &\leq C\left\|\boldsymbol{u}_h\right\|_{L^{3q}}^{1-\vartheta_3}\left\|\triangle\boldsymbol{u}_h\right\|_{L^2}^{\vartheta_3}\cdot\left\|\partial_3\boldsymbol{u}\right\|_{L^q}\cdot\left\|u_3\right\|_{L^{2\lambda}}^{(2\lambda-1)(1-\vartheta_4)}\left\|u_3\right\|_{L^{(2\lambda+1)q}}^{(2\lambda-1)\vartheta_4}\\
&\quad + C\left\|\partial_3\boldsymbol{u}\right\|_{L^q}\left\||u_3|^{\lambda}\right\|_{L^{\frac{2q}{q-1}}}^2,
\end{aligned}$$

where

$$\frac{1}{c}=(1-\vartheta_3)\frac{1}{3q}+\vartheta_3\left(-\frac{2}{3}+\frac{1}{2}\right),\quad \frac{1}{d}=\frac{1-\vartheta_4}{2\lambda}+\frac{\vartheta_4}{(2\lambda+1)q},\quad 0\leq\vartheta_3,\ \vartheta_4\leq 1. \tag{31}$$

By (6) and Lemma 3,

$$
\begin{aligned}
L \leq & C \|\nabla \boldsymbol{u}_h\|_{L^2}^{\frac{2(1-\vartheta_3)}{3}} \|\partial_3 \boldsymbol{u}\|_{L^q}^{\frac{1-\vartheta_3}{3}} \cdot \|\triangle \boldsymbol{u}_h\|_{L^2}^{\vartheta_3} \|\partial_3 \boldsymbol{u}\|_{L^q} \|u_3\|_{L^{2\lambda}}^{(2\lambda-1)(1-\vartheta_4)} \\
& \cdot \left\|\nabla |u_3|^\lambda\right\|_{L^2}^{\frac{2(2\lambda-1)\vartheta_4}{2\lambda+1}} \|\partial_3 u_3\|_{L^q}^{\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}} + C \|\partial_3 \boldsymbol{u}\|_{L^q} \cdot \left\||u_3|^\lambda\right\|_{L^2}^{\frac{2q-3}{q}} \left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{3}{q}} \\
\leq & C \|\nabla \boldsymbol{u}_h\|_{L^2}^{\frac{2(1-\vartheta_3)}{3}} \|\partial_3 \boldsymbol{u}\|_{L^q}^{\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}} \|\triangle \boldsymbol{u}_h\|_{L^2}^{\vartheta_3} \left\||u_3|^\lambda\right\|_{L^2}^{\frac{(2\lambda-1)(1-\vartheta_4)}{\lambda}} \\
& \cdot \left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{2(2\lambda-1)\vartheta_4}{2\lambda+1}} + C \|\partial_3 \boldsymbol{u}\|_{L^q} \cdot \left\||u_3|^\lambda\right\|_{L^2}^{\frac{2q-3}{q}} \left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{3}{q}}.
\end{aligned} \tag{32}
$$

Putting (32) into (28), and integrating in time yields

$$
\begin{aligned}
\mathcal{L}^2(t) \leq & C + C\mathcal{J}^{\frac{2(1-\vartheta_3)}{3}}(t)\mathcal{L}^{\frac{(2\lambda-1)(1-\vartheta_4)}{\lambda}}(t) \cdot \int_\Gamma^t \|\partial_3 \boldsymbol{u}\|_{L^q}^{\frac{1-\vartheta_3}{q}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}} \|\triangle \boldsymbol{u}_h\|_{L^2}^{\vartheta_3} \\
& \left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{2(2\lambda-1)\vartheta_4}{2\lambda+1}} \mathrm{d}\tau + C\mathcal{L}^{\frac{2q-3}{q}}(t) \cdot \int_\Gamma^t \|\partial_3 \boldsymbol{u}\|_{L^q} \left\|\nabla(|u_3|^\lambda)\right\|_{L^2}^{\frac{3}{q}} \mathrm{d}\tau \\
\leq & C + C\tilde{\varepsilon}^{\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}} \mathcal{J}^{\frac{2(1-\vartheta_3)}{3}+\vartheta_3}(t)\mathcal{L}^{\frac{(2\lambda-1)(1-\vartheta_4)}{\lambda}+\frac{2(2\lambda-1)\vartheta_4}{2\lambda+1}}(t) + C\tilde{\varepsilon}\mathcal{L}^2(t),
\end{aligned} \tag{33}
$$

where just as in (26), Hölder inequality with

$$
\frac{2q-3}{2q}\left[\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}\right]+\frac{\vartheta_3}{2}+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}=1 \tag{34}
$$

is applied.

2.3 Closing estimate

By (26) and (33), we have

$$
\mathcal{J}^2(t) \leq C + C\tilde{\varepsilon}\mathcal{J}^2(t) + C\tilde{\varepsilon}^{\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2}\mathcal{J}^{j_1}(t)\mathcal{L}^{l_1}(t), \tag{35}
$$

$$
\mathcal{L}^2(t) \leq C + C\tilde{\varepsilon}^{\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}}\mathcal{J}^{j_2}(t)\mathcal{L}^{l_2}(t) + C\tilde{\varepsilon}\mathcal{L}^2(t), \tag{36}
$$

where

$$
\begin{aligned}
j_1 &= 1+\vartheta_2, & l_1 &= \frac{1-\vartheta_1}{\lambda}+\frac{2\vartheta_1}{2\lambda+1}, \\
j_2 &= \frac{2(1-\vartheta_3)}{3}+\vartheta_3, & l_2 &= \frac{(2\lambda-1)(1-\vartheta_4)}{\lambda}+\frac{2(2\lambda-1)\vartheta_4}{2\lambda+1},
\end{aligned} \tag{37}
$$

and $0 \le \vartheta_i \le 1$ $(1 \le i \le 4)$ should satisfy

$$\begin{aligned}
&\left[\frac{1-\vartheta_1}{2\lambda}+\frac{\vartheta_1}{(2\lambda+1)q}\right]+\left[\frac{1-\vartheta_2}{q}+\vartheta_2\left(-\frac{1}{3}+\frac{1}{2}\right)\right]=\frac{1}{2},\\
&\frac{\vartheta_1}{2\lambda+1}+\frac{2q-3}{2q}\left(\frac{\vartheta_1}{2\lambda+1}+1-\vartheta_2\right)+\frac{1+\vartheta_2}{2}=1,\\
&\left[(1-\vartheta_3)\frac{1}{3q}+\vartheta_3\left(-\frac{2}{3}+\frac{1}{2}\right)\right]+\frac{1}{q}+(2\lambda-1)\left[\frac{1-\vartheta_4}{2\lambda}+\frac{\vartheta_4}{(2\lambda+1)q}\right]=1,\\
&\frac{2q-3}{2q}\left[\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}\right]+\frac{\vartheta_3}{2}+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}=1,
\end{aligned}\tag{38}$$

in view of (21), (22), (27), (30), (31) and (34).

After tedious calculations, we can solve (38) as

$$\begin{aligned}
\vartheta_1&=\frac{(2\lambda-3)(2\lambda+1)(3-q)}{2\lambda q+3q+3\lambda-9}, \quad \vartheta_2=3\frac{\lambda(3-2q)+(5q-6)}{2\lambda q+3q+3\lambda-9},\\
\vartheta_3&=\frac{4\lambda(q+1)-(10q+3)}{2\lambda q-q+\lambda-3}, \quad \vartheta_4=\frac{(2\lambda+1)[9+(3-2\lambda)q]}{3(2\lambda-1)(2\lambda q-q+\lambda-3)}.
\end{aligned}\tag{39}$$

Reducing $0 \le \vartheta_i \le 1$ $(i = 1, 2, 3, 4)$ yields

$$\begin{array}{lllll}
\text{if } \lambda \in & \left[\frac{19}{10}, \frac{33}{16}\right] & \left[\frac{33}{16}, \frac{25+2\sqrt{37}}{18}\right] & \left[\frac{25+2\sqrt{37}}{18}, \frac{5}{2}\right] & \left[\frac{5}{2}, 3\right]\\
\text{then } q \in & \left[\frac{39-6\lambda}{16(\lambda-1)}, \frac{4\lambda-3}{10-4\lambda}\right] & \left[\frac{39-6\lambda}{16(\lambda-1)}, 3\right] & \left[\frac{3(4\lambda-5)}{2(2\lambda-1)}, 3\right] & \left[\frac{3\lambda}{9-2\lambda}, 3\right]
\end{array}\tag{40}$$

respectively. Consequently, if

$$\lambda=\lambda_0\equiv\frac{25+2\sqrt{37}}{18}\approx 2.06475,\tag{41}$$

the range of q is the largest one:

$$q\in\left[\frac{3\sqrt{37}}{4}-3, 3\right].\tag{42}$$

By (39), (37) becomes

$$\begin{aligned}
j_1&=\frac{4\lambda(3-q)+9(2q-3)}{2\lambda q+3q+3\lambda-9}, \quad l_1=\frac{4q-3}{2\lambda q+3q+3\lambda-9},\\
j_2&=\frac{(2\lambda-3)(4q+3)}{3(2\lambda q-q+\lambda-3)}, \quad l_2=\frac{21+10q-6\lambda(2q+1)}{3(2\lambda q-q+\lambda-3)}.
\end{aligned}\tag{43}$$

When (41) and (42) holds, it is obvious that $1 \leq j_1 < 2$, and we may apply Hölder inequality to (35),

$$\begin{aligned}\mathcal{J}^2(t) &\leq C + C\tilde{\varepsilon}\mathcal{J}^2(t) + \frac{1}{2}\mathcal{J}^2(t) + C\mathcal{L}^{\frac{2l_1}{2-j_1}}(t) \\ &\leq C + C\tilde{\varepsilon}\mathcal{J}^2(t) + \frac{1}{2}\mathcal{J}^2(t) + +C\mathcal{L}^{\frac{2}{2\lambda-3}}(t).\end{aligned} \tag{44}$$

Now choose $0 < \tilde{\varepsilon} \ll 1$ sufficiently small such that

$$C\tilde{\varepsilon} \leq \frac{1}{4}, \tag{45}$$

we have

$$\mathcal{J}(t) \leq C + C\mathcal{L}^{\frac{1}{2\lambda-3}}(t). \tag{46}$$

Plugging (46) into (36), and choosing $\tilde{\varepsilon}$ such that

$$C\tilde{\varepsilon}^{\frac{1-\vartheta_3}{3}+1+\frac{(2\lambda-1)\vartheta_4}{2\lambda+1}} \leq \frac{1}{4}, \tag{47}$$

besides (45), we find

$$\mathcal{L}^2(t) \leq C + \frac{1}{4}\mathcal{L}^{\frac{1}{2\lambda-3}j_2+l_2}(t) + \frac{1}{2}\mathcal{L}^2(t) = C + \frac{3}{4}\mathcal{L}^2(t) \Rightarrow \mathcal{L}^2(t) \leq C. \tag{48}$$

Combining (46) and (48), we see that $\|\nabla \boldsymbol{u}_h(t)\|_{L^2}$ is uniformly bounded on $t \in [\Gamma, \Gamma^*)$ as desired. The proof of Theorem 2 is completed.

Remark 4 Cao [2] took $\lambda = 2$ to deduce (8), which corresponds to the range of q in case $\lambda = 2$ in (40). In our paper, we treat all the possibilities to make the range of q as large as possible. The method involves a generalized multiplicative Sobolev inequality (see Lemma 3) and the general $L^{2\lambda}$ estimate, but not just L^4 estimate. For some applications of general $L^{2\lambda}$ estimates, we refer to [10,32], which improves [5].

Acknowledgements This work is supported by the National Natural Science Foundation of China (Grant No. 11501125).

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